# A Long-Short-Term Mixed-Integer Formulation for Highway Lane Change Planning

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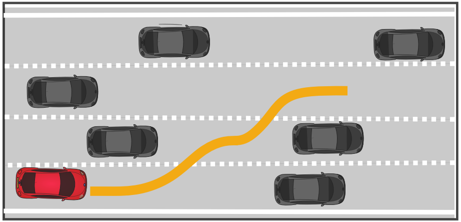
# Outline

1. **Introduction and Preliminaries**
   - **Problem statement**
   - Basic approach: MIQP
   - Efficient MIQP formulations
   - Disjunctive programming
   - Chebeychev centering
2. Method
3. Experiments and Results

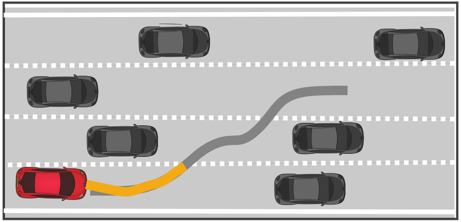
# Highway Motion Planning

**High-Level Planning**



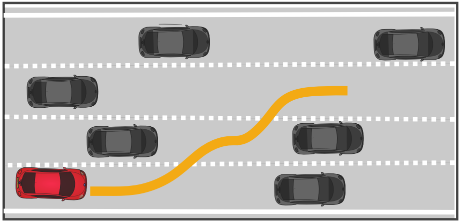
- Behavior planning & decision making

**Low-Level Control**



- Tracking and stabilization

**High-Level Planning**



- Behavior planning & decision making
- Lower frequency & larger horizon
- Low fidelity model

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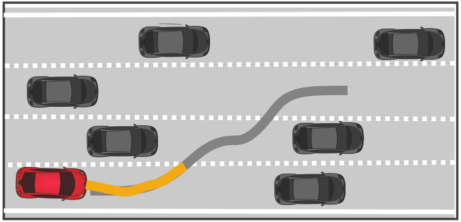
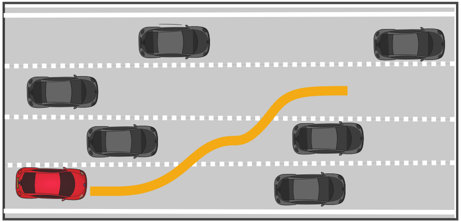


- Tracking and stabilization
- Higher frequency & shorter horizon
- Medium to high fidelity model

# Highway Motion Planning

## High-Level Planning



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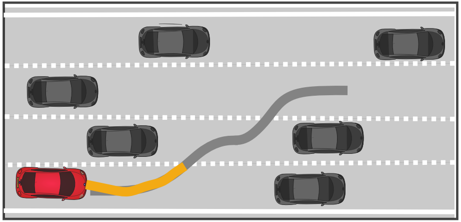

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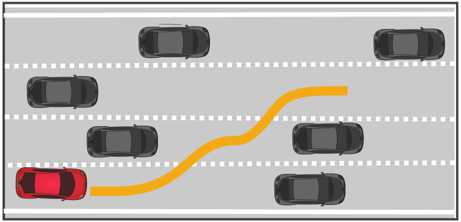
# Highway Motion Planning

### High-Level Planning



- Behavior planning & decision making
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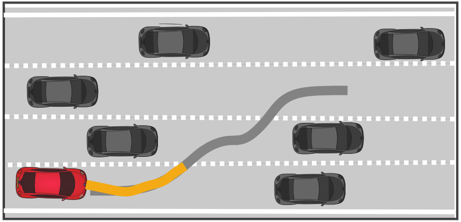
**Algorithm**

- Mixed-integer optimization

### Low-Level Control



- Tracking and stabilization
- Higher frequency & shorter horizon
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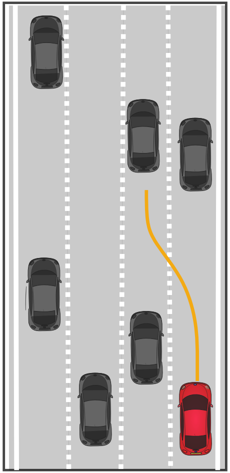
**Algorithm**

- Sequential quadratic programming

# Outline

# High-Level Planning Utilizing Mixed-Integer Optimization

Baseline[1,2]



## Objective

High-level planning with obstacles

## Baseline[2]

MIQP framework requiring

- Quadratic objective $\tilde{(\cdot)}$
- Linear constraints $(\cdot)'$
- Binary variables for nonconvex constraints

## Properties

+ MIQP solver can find global optimum
– High computational burden

With $\beta_k \in \{0,1\}^{4N_{\mathrm{obs}}}$,

$$\begin{aligned}
\min_{\substack{u_0,\dots,u_{N-1}\\ x_0,\dots,x_N\\ \beta_0,\dots,\beta_N}} \quad & \sum_{k=0}^{N-1} \tilde{l}(x_k,u_k)+\tilde{J}(x_N)\\
\text{s.t.} \quad & x_0=\hat{x},\\
& 0 \geq h'_{\mathrm{t,sv}}(x_N,\beta_N;\hat{\tau}_{\mathrm{sv}}),\\
& 0 \geq h'_{\mathrm{t,model}}(x_N),\\
& x_{k+1}=F'(x_k,u_k),\\
& 0 \geq h'_{\mathrm{sv}}(x_k,\beta_k;\hat{\tau}_{\mathrm{sv}}),\\
& 0 \geq h'_{\mathrm{model}}(x_k,u_k),\\
& k=0,\dots,N-1.
\end{aligned}$$

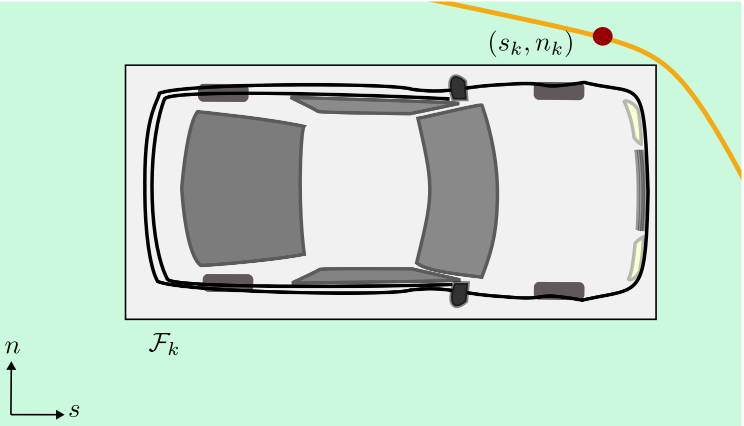
[1] Qian et al., 2016.

[2] Quirynen, Safaoui, and Di Cairano, 2023.

# High-Level Planning Utilizing Mixed-Integer Optimization

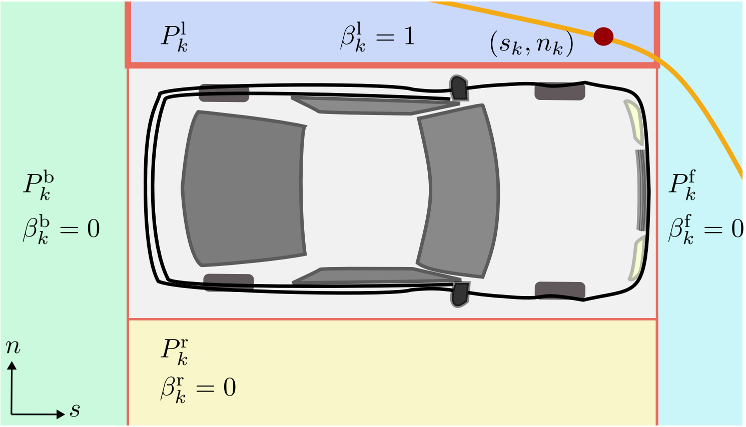
Collision constraints with binary variables

**Obstacle avoidance leads to nonconvex constraints**




Feasible set $\mathcal{F}$ (green) around obstacle vehicle at time step $k$.

**Decomposition into convex sets $P_k^{\{\cdot\}}$ linked to binary variables**

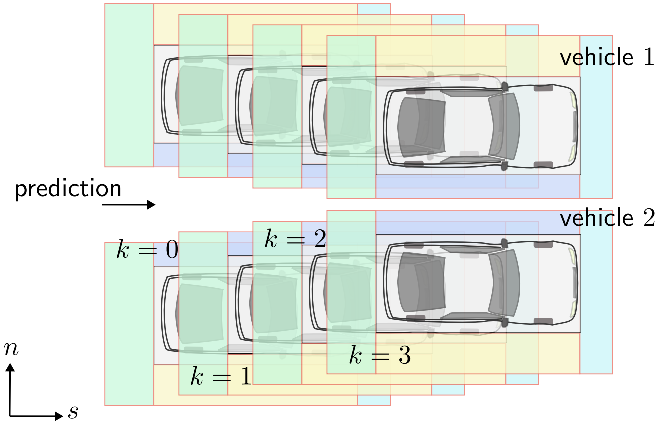



Four binary variables $\beta_k^{\{\cdot\}}$ are used to formulate disjunctive constraints.

# High-Level Planning Utilizing Mixed-Integer Optimization

## Collision constraints with binary variables

### Decomposition for each prediction step and vehicle




Binary variable required for each obstacle and each time step: $N_{\text{bin}} = 4NN_{\text{obs}}$.

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# Efficient MIQP Formulations for High-Level Planning

To achieve speedup of baseline[3,4,5]

Baseline: MIQP-based obstacle avoidance. Number of binary variables $N_{\text{bin}} = 4N_{\text{obs}}N$.

**Approach Decoupling[3]**

Decomposition MILP - NLP

**Application**

- Static obstacles

**Features**

- + $N_{\text{bin}} = N_{\text{obs}}$
- + Real-time feasible (<2 s)

[3] Reiter et al., 2021.

[4] Reiter et al., 2024a.

[5] Reiter et al., 2024b.

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### Approach 2-Horizon[4]

Long & short horizon with coupling in single MIQP

**Application**

- Highways

**Features**

+ $N_{\text{bin}} = \mathcal{O}(N_{\text{obs}} + N)$
+ Speedup: 2 to 100 times

[3] Reiter et al., 2021.

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# Efficient MIQP Formulations for High-Level Planning

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### Approach Learning[5]

NN to predict binary variables of MIQP

**Application**

- Generic high-level planning (equal to baseline)

**Features**

+ $N_{\text{bin}} = 0$
+ Speedup $> 100$ times
– Requires safety framework

[3] Reiter et al., 2021.
[4] Reiter et al., 2024a.
[5] Reiter et al., 2024b.

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# Product Linearization and Big-M Formulations

For a given compact set $\mathcal{X} \subset \mathbb{R}$ and continuous function $f : \mathcal{X} \to \mathbb{R}$, let $\overline{M} \geq \max_{x \in \mathcal{X}} f(x)$ and $\underline{M} \leq \min_{x \in \mathcal{X}} f(x)$ denote an upper and lower bound of $f(x)$ on $\mathcal{X}$, respectively. The following properties hold for a given $f : \mathcal{X} \to \mathbb{R}$.
For a product $y = \beta f(x)$, with $y \in \mathbb{R}$, the following equivalence holds for all $x \in \mathcal{X}$ and $\beta \in \mathbb{N}_{[0:1]}$:

$$y = \beta f(x) \Leftrightarrow \begin{cases} y & \leq \overline{M}\beta, \\ y & \geq \underline{M}\beta, \\ y & \leq f(x) - \underline{M}(1-\beta), \\ y & \geq f(x) - \overline{M}(1-\beta). \end{cases}$$

The implications $[\beta = 1] \implies [f(x) \geq 0]$ and $[f(x) < 0] \implies [\beta = 0]$ of a binary variable $\beta \in \mathbb{N}_{[0:1]}$ that activates constraint $f(x) \geq 0$, is formulated as

$$f(x) \geq \underline{M}(1-\beta).$$

The implications $[\beta = 0] \implies [f(x) \leq 0]$ and $[f(x) > 0] \implies [\beta = 1]$ of a binary variable $\beta \in \mathbb{N}_{[0:1]}$ that gets activated if constraint $f(x) > 0$ is valid, is formulated as

$$f(x) \leq \overline{M}\beta.$$

# Formulating Disjunctions within MIQPs

The disjunction $\bigvee_{i=1}^{N}[f_i(x) \geq 0]$ is formulated by adding $N$ binary variables $\beta_i \in \mathbb{N}_{[0:1]}$ with $i \in \mathbb{N}_{[1:N]}$, and the conditions

$$[\beta_i = 1] \implies [f_i(x) \geq 0], \forall i \in \mathbb{N}_{[1:N]}$$

$$\sum_{i=1}^{N} \beta_i \geq 1$$

For an exclusive disjunction, it is required that

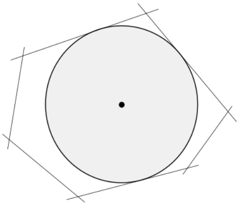
$$[\beta_i = 1] \Leftrightarrow [f_i(x) \geq 0], \forall i \in \mathbb{N}_{[1:N]}$$

$$\sum_{i=1}^{N} \beta_i = 1$$

# Outline

The Chebychev center (CC) of a polyhedron $P = \{x \mid Ax \leq b\}$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, is the center $x^\star \in \mathbb{R}^n$ of the largest ball $B(x^\star, r^\star) = \{x \mid \|x^\star - x\| \leq r^\star\}$ contained in $P$. The radius $r^\star$ is called the Chebychev radius. With $A_i$ and $b_i$ being the $i$-th row of $A$ and $b$, respectively, the CC and Chebychev radius can be computed by solving the linear program

$$
\begin{aligned}
\min_{r,\,x} \quad & -r \\
\text{s.t.} \quad & A_i x + r\|A_i\|_2 \leq b_i \quad i \in \mathbb{N}_{[1:m]} \\
& r \geq 0
\end{aligned}
$$



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1. Introduction and Preliminaries
2. **Method**
   - **Overview**
   - Short-term formulation
   - Long-term formulation
   - Coupling
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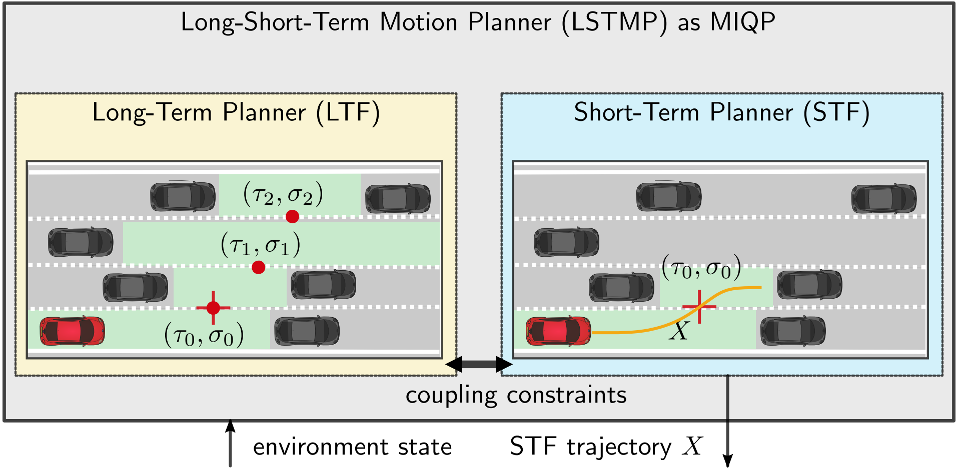
# High-Level Overview

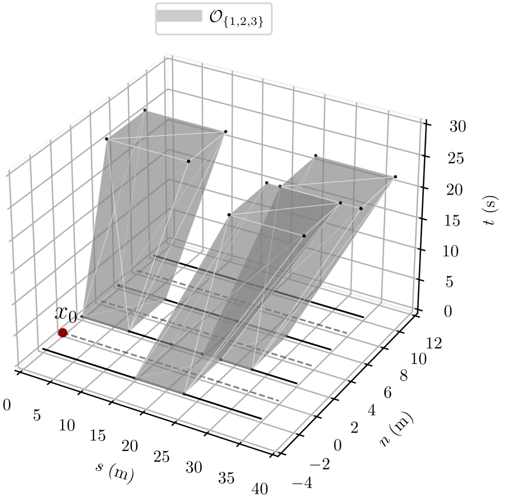
Separation of the planning problem in two horizons:
(i) Geometric abstraction for **many transitions** (LTF), (ii) linear model for **first transition** (STF)

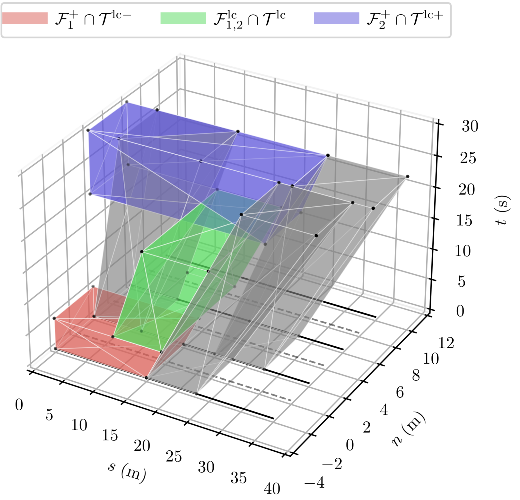


# Outline

# Short-Term Formulation (STF)

## Convex sets for transition into next lane



# Short-Term Formulation (STF)

## Convex sets for transition into next lane

- Linear model for vehicle dynamics: $x_{k+1} = Ax_k + Bu_k$
- Using binary variables $\lambda_k$ to define lateral reference $\tilde{n}_k$. The STF models a maximum of one transition to the next lane.

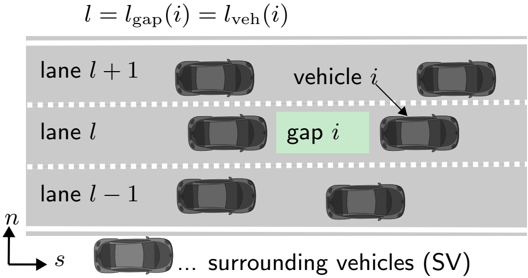
$$\tilde{n}_k = d_{\text{lane}} \lambda_k, \ k \in \mathbb{N}_{[0:N]}, \lambda_{k+1} \geq \lambda_k, \ k \in \mathbb{N}_{[0:N-1]}.$$

- Transition follows three stages w.r.t. lane change duration $\bar{t}_{\text{lc}}$ and transition time $\tau$
  1. The ego lane is tracked for $t \leq \tau_1 - \frac{1}{2}\bar{t}_{\text{lc}}$. States constrained to $\mathcal{F}_1^+$
  2. Transition to the next lane for $\tau_1 + \frac{1}{2}\bar{t}_{\text{lc}} > t > \tau_1 - \frac{1}{2}\bar{t}_{\text{lc}}$. States constrained to $\mathcal{F}_{1,g+}^{\text{lc}}$
  3. The next lane is tracked for $t \geq \tau_1 + \frac{1}{2}\bar{t}_{\text{lc}}$. States constrained to $\mathcal{F}_{g+}^+$
- We use binary variables $\lambda$ to constrain transitions to sets, with $n_{\text{lc}} = \lceil \frac{\bar{t}_{\text{lc}}}{2t_{\text{d}}} \rceil$ and

$$\begin{aligned}
[1 - \lambda_{k+n_{lc}} = 1] &\implies (t_k, s_k, n_k) \in \mathcal{F}_1^+, \\
[\lambda_{k+n_{lc}} - \lambda_{k-n_{lc}} = 1] &\implies (t_k, s_k, n_k) \in \mathcal{F}_{1,g+}^{\text{lc}}, \\
[\lambda_{k-n_{lc}} = 1] &\implies (t_k, s_k, n_k) \in \mathcal{F}_{g+}^+,
\end{aligned}$$

# Outline

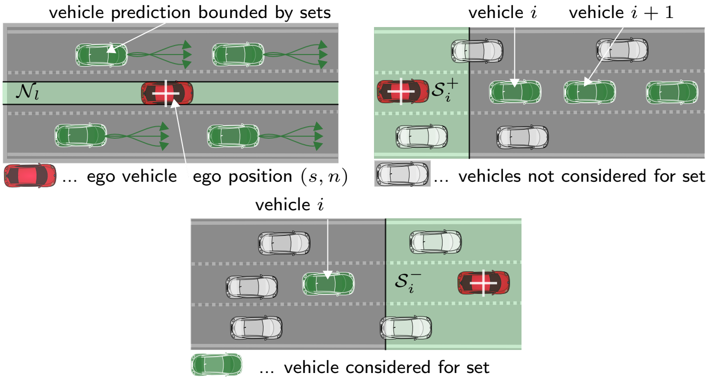
- Surrounding vehicles (SVs) and gaps are uniquely enumerated
- We want to look at how free spaces evolve over time?



# Long-Term Formulation (LTF)

## Free spaces defined by intersection of hyperplanes

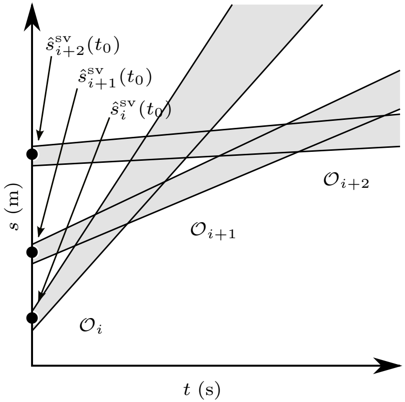
- Sets $\mathcal{N}_l, \mathcal{S}_i^+$ and $\mathcal{S}_i^-$ define half-spaces in the spatio-lateral-temporal (SLT)-space
- Free set $\mathcal{S}_i^+$ rear to vehicle $i$
- Free set $\mathcal{S}_i^-$ in front of vehicle $i$



# Long-Term Formulation (LTF)

## Position-time (ST) view

- Surrounding vehicles (SVs) $i, i+1$ and $i+2$ are assumed to move within cones $\mathcal{O}_i$, $\mathcal{O}_{i+1}$ and $\mathcal{O}_{i+2}$ in position-time-space (ST space) on single lane (if not blocked).

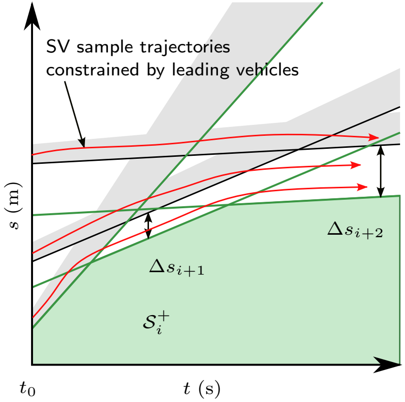



Occupied ST space of three SVs

# Long-Term Formulation (LTF)

## Construction of obstacle-free sets $\mathcal{S}_i^+$

Longitudinal obstacle-free space $\mathcal{S}_i^+$ for an SV with index $i$ is constructed by also considering slower leading SVs. Red trajectories correspond to samples of possible SV trajectories.

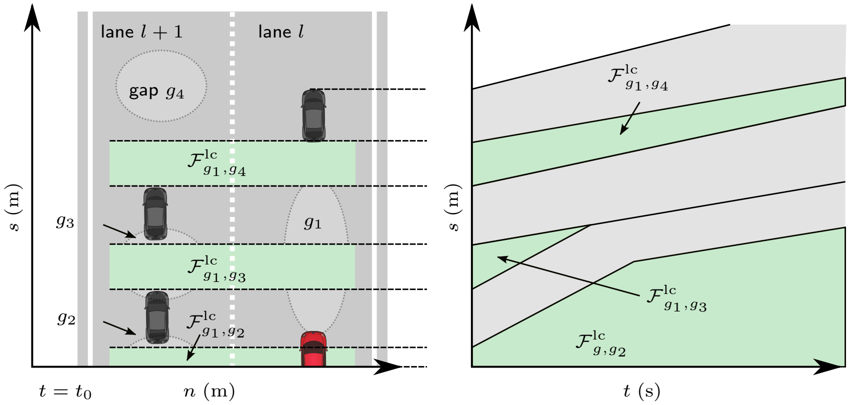



Free space $\mathcal{S}_i^+$ includes linear shifted constraints of all leaders.

# Long-Term Formulation (LTF)

Combining free longitudinal spaces to guarantee safe lane changes

- Safe transition from lane $l$ to $l+1$ → intersection of individual free spaces to obtain $\mathcal{F}^{\mathrm{lc}}$ (green)
- Three gaps $g_2$, $g_3$, and $g_4$ on the consecutive lane $l+1$ are available to transition from gap index $g_1$ and lane $l$

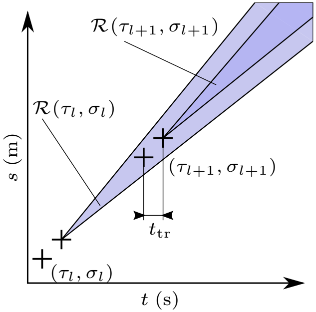



Possible gaps in Frenet coordinates (left) and ST coordinates (right).

# Long-Term Formulation (LTF)

## Reachable set in the ST-space

- Next, we consider approximate the possible ego trajectories by cones $\mathcal{R}$
- This is the most "severe" approximation, usually reachable sets of point masses would be nonconvex, quadratic in one dimension
- We assume cone shifted by traversal time $t_{\mathrm{tr}}$ from last transition (time $\tau_l$, lon. position $\sigma_l$)

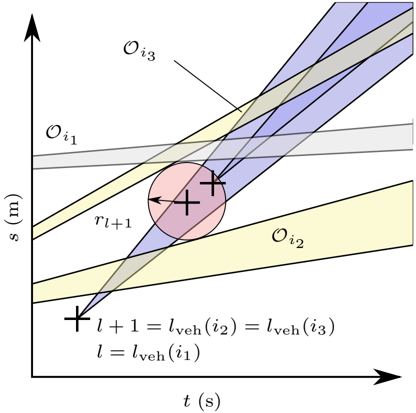



Reachable on lane $l$ and after transition on lane $l+1$

# Long-Term Formulation (LTF)

## Bringing all together (CC centering, reachable set, SV sets)

- Looking for the transition $\tau_l$, $\sigma_l$ that is farthest away from obstacles
- Using Chebychev center (CC) centering to find transition and constrain center to reachable set




CC (red) of the transition to lane $l+1$ from gap $i_1$ to gap $i_3$ with two SVs on the next lane $l+1$ (yellow) and one SV on the current lane $l$ (grey).

- Continuous decision variables for LTF:
  - transition times $T = [\tau_1, \ldots, \tau_{L-1}]^\top$
  - longitudinal transition positions $\Sigma = [\sigma_1 \ldots, \sigma_{L-1}]^\top$
- Reachable set constraint

$$\mathcal{R}(\tau_l, \sigma_l) = \left\{ (\tau_{l+1}, \sigma_{l+1}) \middle| \begin{matrix} \sigma_{l+1} \leq \sigma_l + \overline{v}_{\mathrm{op}}(\tau_{l+1} - \tau_l - \tilde{t}_{\mathrm{lc}}) \\ \sigma_{l+1} \geq \sigma_l + \underline{v}_{\mathrm{op}}(\tau_{l+1} - \tau_l + \tilde{t}_{\mathrm{lc}}) \end{matrix} \right\}.$$

Chebychev centering within a given sequence of gaps

- Sequence of gap indices $[g_1, \ldots, g_{L-1}]$
- Transition radii $R = [r_1, \ldots, r_{L-1}]^\top$
- Chebeychev centering of $[\tau,\ \sigma] \in \mathcal{S}_g^+$ written as $h_g^+(\tau, \sigma, r) \leq 0$
- Chebeychev centering of $[\tau,\ \sigma] \in \mathcal{S}_g^- \cap \mathcal{S}_g^+$ written as $h_g(\tau, \sigma, r) \leq 0$
- Maximizing the sum of all Chebychev radii, given a sequence of gaps

$$\min_{R, \Sigma, T} \quad - w_{\text{safe}} \sum_{l=1}^{L-1} r_l \tag{1a}$$

$$\text{s.t.} \quad h_{g_l}^+(\tau_l, \sigma_l, r_l) \leq 0, \quad l \in \mathbb{N}_{[1:L-2]} \tag{1b}$$

$$h_{g_{l+1}}(\tau_l, \sigma_l, r_l) \leq 0, \quad l \in \mathbb{N}_{[1:L-2]} \tag{1c}$$

$$\underline{r} \leq r_l, \quad l \in \mathbb{N}_{[1:L-1]}. \tag{1d}$$

# Long-Term Formulation (LTF)

## Disjunctive programming in the ST-space

Disjunctive formulation among all gaps within a lane, including an extra virtual gap to avoid transition

- Set of all gap indices on a particular lane $\mathcal{G}_l := \{g \mid l = l_{\text{gap}}(g)\}$
- Disjunctions of constraints among "next lane gaps"

$$\bigvee_{g^+ \in \mathcal{G}_{l+1}} \begin{bmatrix} \beta_{g^+} \\ h_{g^+}(\tau_l, \sigma_l, r_l) \leq 0 \end{bmatrix}, \quad \forall l \in \mathbb{N}_{[1:L-1]},$$

- Disjunctions of constraints among "current lane gaps"

$$\bigvee_{g \in \mathcal{G}_l} \begin{bmatrix} \beta_g \\ h_g^+(\tau_l, \sigma_l, r_l) \leq 0 \end{bmatrix}, \quad \forall l \in \mathbb{N}_{[2:L-1]},$$

- On the current lane $l$, related to transition $l$ only leading vehicle constraints are considered
- On the next lane $l+1$ related to transition $l$ leading and following vehicles on lane $l+1$ are considered
- For each pair of consecutive virtual gaps $\hat{g}$ and $\hat{g}^+$, with $l(\hat{g}^+) = l(\hat{g}) + 1$, we require $\beta_{\hat{g}^+} \geq \beta_{\hat{g}}$

# Outline

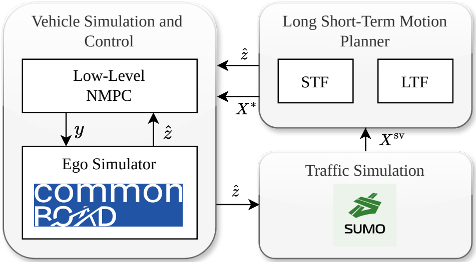
# Coupling of LTF and STF

Relating the first transition point of the LTF $(\tau_1, \sigma_1)$ to the STF states $(t_k, s_k)$. The STF already constrained the lateral state $n_k$.

$$\begin{bmatrix} [\lambda_k = 1] \\ k t_d \geq \tau_1 \\ s_k \geq \sigma_1 \end{bmatrix} \vee \begin{bmatrix} [\lambda_k = 0] \\ k t_d < \tau_1 \\ s_k < \sigma_k \end{bmatrix}, \quad \forall k \in \mathbb{N}_{[1:N]},$$

# Outline

1. Deterministic simulation with only braking interaction
2. Randomized SUMO scenarios with interactive cars



# Experiments and Results

Pros:

- Proposed LSTMP is up to 100x faster (10-40ms) than MIQP baseline[6]
- Computation time does not scale with prediction horizon
- Superior closed-loop cost compared to hybrid $A^\star$ and MIQP baseline

Cons:

- Limited to unidirectional lane changes compared to MIP-DM
- Limited to highway scenarios
- Approximation of reachable set potentially causes recursive infeasibility (but always safe due STF)

Outlook:

- If we would allow MINLP formulations (nonconvex quadratic reachable sets), we would get rid of STF, be recursive feasible, more accurate and achive a concise problem formulation
- Test MINLP solver from Andrea?
- Probably there exists a similar bidirectional formulation

[6] Quirynen, Safaoui, and Di Cairano, 2023.

Thank you for your attention!